

Characterization of the electric field concentration between two adjacent spherical perfect conductors*

Hyeonbae Kang[†]

Mikyoung Lim[‡]

KiHyun Yun[§]

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Abstract

When two perfectly conducting inclusions are located closely to each other, the electric field concentrates in a narrow region in between two inclusions, and becomes arbitrarily large as the distance between two inclusions tends to zero. The purpose of this paper is to derive an asymptotic formula of the concentration which completely characterizes the singular behavior of the electric field, when inclusions are balls of the same radii in three dimensions.

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1 Introduction and statement of results

Let D_1 and D_2 be bounded, simply connected and convex domains in \mathbb{R}^d , $d = 2, 3$. Suppose that the conductivity of the inclusions is ∞ , in other words, inclusions are perfect conductors. We consider the following conductivity problem:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D_1 \cup D_2}, \\ u = C_j \text{ (constant)} & \text{on } \partial D_j, j = 1, 2, \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty, \end{cases} \quad (1.1)$$

where H is a given harmonic function in \mathbb{R}^d so that $-\nabla H$ is the background electric field in the absence of the inclusions. The constant value C_j on ∂D_j is determined by the condition

$$\int_{\partial D_j} \frac{\partial u}{\partial \nu^{(j)}} d\sigma = 0 \quad \text{for } j = 1, 2. \quad (1.2)$$

Here and throughout this paper $\nu^{(j)}$ is the outward unit normal to ∂D_j .

The gradient of the solution ∇u represents the electric field (with the opposite sign) in the presence of inclusions and the stress field in two dimensional anti-plane elasticity, and it may become arbitrarily large as the distance between two inclusions tends to 0. It has been proved that the generic rate of the gradient blow-up is $\epsilon^{-1/2}$ in two dimensions [2, 4, 5, 8, 11, 14, 15] and $|\epsilon \log \epsilon|^{-1}$ in three dimensions [5, 6, 12, 13], where ϵ is the distance between two inclusions. Occurrence of the gradient blow-up depends on the background potential (the harmonic function H in (1.1)) and

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[†]Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea (hbkang@inha.ac.kr)

[‡]Department of Mathematics, Korea Advanced Institute of Science and Technology, Yuseong-gu, Daejeon 305-701, Republic of Korea (mklm@kaist.ac.kr)

[§]Department of Mathematics, Hankuk University of Foreign Studies, Youngin-si, Gyeonggi-do 449-791, Republic of Korea (gundam@hufs.ac.kr)

those background potentials which actually make the gradient blow up are characterized in [3] when D_1 and D_2 are disks.

The results mentioned above are estimates of the gradient of the solution from above and below, namely,

$$\frac{C_1}{\psi(\epsilon)} \leq |\nabla u| \leq \frac{C_2}{\psi(\epsilon)} + C_3 \quad (1.3)$$

for some positive constants C_1 , C_2 and C_3 where

$$\psi(\epsilon) = \begin{cases} \sqrt{\epsilon} & \text{if } d = 2, \\ \epsilon \log \frac{1}{\epsilon} & \text{if } d = 3. \end{cases} \quad (1.4)$$

The constants C_1 and C_2 can possibly be 0 depending on the background potential H .

The interest of this paper lies in the asymptotic behavior of ∇u as the distance between two inclusions tends to 0. Since the singular behavior of ∇u occurs in the narrow region in between two inclusions, we are particularly interested in its behavior there. In this regards, a complete characterization of the singular behavior of ∇u has been obtained when inclusions are disks [10] and strictly convex domains in \mathbb{R}^2 [1]. Let D_1 and D_2 be disks in \mathbb{R}^2 of radii r_1 and r_2 , respectively, and let R_j be the reflection with respect to ∂D_j , $j = 1, 2$. Then the combined reflections $R_1 R_2$ and $R_2 R_1$ have unique fixed points, say $\mathbf{f}_1 \in D_1$ and $\mathbf{f}_2 \in D_2$. Let

$$h(\mathbf{x}) = \frac{1}{2\pi} (\log |\mathbf{x} - \mathbf{f}_1| - \log |\mathbf{x} - \mathbf{f}_2|) \quad (1.5)$$

(see section 2 for a discussion on the function h). It has been proved that the solution u to (1.1) can be expressed as

$$u(\mathbf{x}) = \frac{4\pi r_1 r_2}{r_1 + r_2} (\mathbf{n} \cdot \nabla H)(\mathbf{c}) h(\mathbf{x}) + g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus (D_1 \cup D_2), \quad (1.6)$$

where \mathbf{c} is the middle point of the shortest line segment connecting ∂D_1 and ∂D_2 , \mathbf{n} is the unit vector in the direction of $\mathbf{f}_2 - \mathbf{f}_1$, and $|\nabla g(\mathbf{x})|$ is bounded independently of ϵ on any bounded subset of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. So the singular behavior of ∇u is completely characterized by ∇h . In particular, it can be shown using (1.6) that the maximal concentration of ∇u occurs along the shortest line segment connecting ∂D_1 and ∂D_2 , and on that segment

$$\nabla u \approx \frac{2\sqrt{2}}{\sqrt{\epsilon}} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} (\mathbf{n} \cdot \nabla H)(\mathbf{c}) \mathbf{n}. \quad (1.7)$$

A complete characterization of the gradient blow-up like (1.6) has been obtained in [1] in the case when inclusions are strictly convex domains in \mathbb{R}^2 by using disks osculating to convex domains. It is worth mentioning that the stress concentration factor for the p -Laplacian was derived in [9].

The purpose of this paper is to derive an asymptotic formula of ∇u which characterizes its singular behavior when D_1 and D_2 are balls of the same radii in three dimensions.

In order to state the main result of this paper in a precise manner, let us fix notation. Let D_1 and D_2 be balls of radius r in three dimensions and \mathbf{c}_1 and \mathbf{c}_2 their centers. Let \mathbf{c} be the middle point of \mathbf{c}_1 and \mathbf{c}_2 , and \mathbf{n} the unit vector in the direction of $\mathbf{c}_2 - \mathbf{c}_1$, *i.e.*,

$$\mathbf{c} = \frac{\mathbf{c}_1 + \mathbf{c}_2}{2}, \quad \mathbf{n} = \frac{\mathbf{c}_2 - \mathbf{c}_1}{|\mathbf{c}_2 - \mathbf{c}_1|}.$$

Let R_j , $j = 1, 2$, be the reflection with respect to ∂D_j , *i.e.*,

$$R_j(\mathbf{x}) = \frac{r(\mathbf{x} - \mathbf{c}_j)}{|\mathbf{x} - \mathbf{c}_j|^2} + \mathbf{c}_j,$$

and let, for $k = 0, 1, \dots$,

$$\begin{cases} \mathbf{p}_{2k} = (R_2 R_1)^k \mathbf{c}_2, \\ \mathbf{p}_{2k+1} = R_2 (R_1 R_2)^k \mathbf{c}_1. \end{cases} \quad (1.8)$$

We emphasize that \mathbf{p}_n is contained in D_2 and monotonically converges to \mathbf{p} as $n \rightarrow \infty$ where \mathbf{p} is the fixed point of the combined reflection $R_2 R_1$. Let

$$\mu_n = \frac{1}{|\mathbf{c}_1 - \mathbf{p}_n|}, \quad n = 1, 2, \dots, \quad (1.9)$$

and

$$q_0 = 1 \quad \text{and} \quad q_n = \prod_{j=1}^n \mu_j, \quad n \geq 1. \quad (1.10)$$

Let $\rho(\mathbf{x})$ be the distance from \mathbf{x} to the line connecting \mathbf{c}_1 and \mathbf{c}_2 , i.e.,

$$\rho(\mathbf{x}) = |(\mathbf{x} - \mathbf{c}) - \langle \mathbf{x} - \mathbf{c}, \mathbf{n} \rangle \mathbf{n}|. \quad (1.11)$$

The following is the main result of this paper.

Theorem 1.1 *Suppose that the radius of the balls is much larger than the distance between them, i.e., $\epsilon \ll r$. The gradient ∇u of the solution to (1.1) can be expressed as*

$$\nabla u(\mathbf{x}) = \frac{C_H^\epsilon}{|\log \epsilon| (\epsilon + r\rho(\mathbf{x})^2)} (\mathbf{n} + \eta(\mathbf{x})) + \nabla g(\mathbf{x}) \quad \text{if } \rho(\mathbf{x}) \leq \frac{r}{|\log \epsilon|^2} \quad (1.12)$$

where

$$C_H^\epsilon = 2 \sum_{n=0}^{\infty} q_n (H(\mathbf{p}_n) - H(-\mathbf{p}_n)), \quad (1.13)$$

$|\nabla g|$ is bounded on any bounded region in $\mathbb{R}^3 \setminus (D_1 \cup D_2)$ regardless of ϵ , and

$$|\eta(\mathbf{x})| \leq C |\log \epsilon|^{-1} \quad (1.14)$$

for some constant $C > 0$ independent of ϵ .

Some remarks on Theorem 1.1 are in order. We first observe that the set $\rho(\mathbf{x}) \leq r |\log \epsilon|^{-2}$ where (1.12) holds is a narrow region in between two spheres. The formula (1.12) shows that the major singular term of ∇u is in the direction of \mathbf{n} , and that if $\rho(\mathbf{x}) = \text{constant}$, then intensity of the field is constant. Note that the level set where $\rho(\mathbf{x})$ is constant is a cylinder around the line connecting centers of two spheres. So the intensity of the field decreases radially from the line connecting two centers of spheres. The highest concentration of the field occurs when $\rho(\mathbf{x}) = 0$, in other words, when \mathbf{x} is on the line segment connecting two closest points on the spheres, and on the segment,

$$\nabla u \approx \frac{C_H^\epsilon}{\epsilon |\log \epsilon|} \mathbf{n}. \quad (1.15)$$

Note that C_H^ϵ depends on ϵ since \mathbf{p}_n and q_n do. The following theorem reveals the limiting behavior of C_H^ϵ as $\epsilon \rightarrow 0$.

Theorem 1.2 *We have*

$$C_H^\epsilon = C_H + O(\sqrt{\epsilon} |\log \epsilon|) \quad \text{as } \epsilon \rightarrow 0 \quad (1.16)$$

where

$$C_H = 2 \sum_{n=0}^{\infty} \frac{1}{n} \left(H\left(\frac{r}{n} \mathbf{n} + \mathbf{c}\right) - H\left(-\frac{r}{n} \mathbf{n} + \mathbf{c}\right) \right). \quad (1.17)$$

In particular, if $\rho(\mathbf{x}) = 0$, then

$$\lim_{\epsilon \rightarrow 0} \epsilon |\log \epsilon| |\nabla u(\mathbf{x})| = |C_H|. \quad (1.18)$$

We emphasize that the occurrence of the gradient blow-up depends on the constant C_H : if $C_H \neq 0$, then it occurs. If $C_H = 0$, then either $|\nabla u|$ is bounded or the blow-up rate is weaker than the generic rate $(\epsilon |\log \epsilon|)^{-1}$. One can show for example that if the centers of the balls lie on the x -axis and their middle point is $(0, 0, 0)$, and if $H(x, y, z) = x^3 - 3xy^2$, then $C_H \neq 0$ and hence $|\nabla u|$ blows up as $\epsilon \rightarrow 0$. It is interesting to observe that this is in contrast with two dimensional circular case. In view of (1.7), the blow-up occurs only when $(\mathbf{n} \cdot \nabla H)(0, 0) \neq 0$ (assuming $\mathbf{c} = (0, 0)$). So, $\nabla u(x, y)$ blows up in two dimensions only when the background potential H has the linear term $\mathbf{n} \cdot \mathbf{x}$.

The main ingredient in deriving (1.12) is the singular function h which is the solution to

$$\begin{cases} \Delta h = 0 & \text{in } \mathbb{R}^d \setminus \overline{D_1 \cup D_2}, \\ h = \text{constant} & \text{on } \partial D_j, \ j = 1, 2, \\ \int_{\partial D_j} \frac{\partial h}{\partial \nu^{(j)}} ds = (-1)^{j+1}, & j = 1, 2, \\ h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (1.19)$$

Such a solution exists and is unique (see [1, 14]). We emphasize that the constant values of h on ∂D_1 and on ∂D_2 are different, and because of that the gradient of h becomes arbitrarily large if the distance between D_1 and D_2 is small. This function characterizes the singular behavior of the solution to (1.1). In fact, if we define the function g by

$$u(\mathbf{x}) = \frac{u|_{\partial D_2} - u|_{\partial D_1}}{h|_{\partial D_2} - h|_{\partial D_1}} h(\mathbf{x}) + g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus (D_1 \cup D_2), \quad (1.20)$$

then one can see that g is harmonic in $\mathbb{R}^d \setminus \overline{D_1 \cup D_2}$ and $g|_{\partial D_1} = g|_{\partial D_2}$, in other words, there is no potential difference of g on ∂D_1 and ∂D_2 . So it can be shown in the same way as in [10] that $|\nabla g|$ is bounded on bounded subsets of $\mathbb{R}^d \setminus (D_1 \cup D_2)$. It means that the singular behavior of ∇u is completely determined by $\frac{u|_{\partial D_2} - u|_{\partial D_1}}{h|_{\partial D_2} - h|_{\partial D_1}} \nabla h(\mathbf{x})$. Moreover, it is proved in [14, 15] that

$$u|_{\partial D_1} - u|_{\partial D_2} = \int_{\partial D_1} H \frac{\partial h}{\partial \nu^{(1)}} d\sigma + \int_{\partial D_2} H \frac{\partial h}{\partial \nu^{(2)}} d\sigma, \quad (1.21)$$

which means that the potential difference of u is determined by the singular function h (and the background potential H).

The function h was first introduced in [14] and used in a crucial way to derive estimates for the gradient blow-up in [13, 14, 15]. It is worth mentioning that $(\frac{\partial h}{\partial \nu^{(1)}}, \frac{\partial h}{\partial \nu^{(2)}})$ is an eigenvector corresponding to the eigenvalue $1/2$ of the Neumann-Poincaré operator associated with the interface problem (1.1) as shown in [1, 7].

If D_1 and D_2 are disks, then h is given by (1.5). In fact, ∂D_1 and ∂D_2 are the Apollonian circles of the fixed points \mathbf{f}_1 and \mathbf{f}_2 , and hence $|\mathbf{x} - \mathbf{f}_1|/|\mathbf{x} - \mathbf{f}_2|$ is constant on ∂D_1 and ∂D_2 . It is worth emphasizing that here the radii of disks may be different. If D_1 and D_2 are spheres, it is proved in [13] that h is given by a weighted sum of the difference of the point charges: let $\Gamma(\mathbf{x}) = \frac{1}{4\pi} |\mathbf{x}|^{-1}$, the fundamental solution of the Laplacian in three dimensions. Then the singular function h is given by

$$h(\mathbf{x}) = \frac{1}{\sum_{n=0}^{\infty} q_n} \sum_{n=0}^{\infty} q_n (\Gamma(\mathbf{x} - \mathbf{p}_n) - \Gamma(\mathbf{x} + \mathbf{p}_n)). \quad (1.22)$$

This formula has been used in [13] to derive estimates like (1.3). We emphasize that in [13] an upper bound for h is derived in a more general case when the radii of spheres are allowed to be different. In this paper we derive finer estimates of h for the purpose of deriving (1.12).

This paper is organized as follows. In section 2, we review the construction of the singular function in [13]. In section 3, we prove some technical lemmas which are required to estimate the singular function. In section 4, we derive an asymptotic formula of the singular function. In the last section, we prove Theorem 1.1 and Theorem 1.2.

2 Singular functions on spheres

Since the radius r is much larger than ϵ , we may assume after scaling if necessary that $r = 1$. We may also assume the centers are on the x -axis and $\mathbf{c} = (0, 0, 0)$ after rotation and shifting if necessary. We assume so in the sequel. It is also convenient to write $\epsilon = 2\delta$ so that $\mathbf{c}_1 = (-1 - \delta, 0, 0)$ and $\mathbf{c}_2 = (1 + \delta, 0, 0)$. Then, the function ρ defined in (1.11) becomes

$$\rho(x, y, z) = \sqrt{y^2 + z^2}, \quad (2.1)$$

and $\mathbf{n} = (1, 0, 0)$. Note that \mathbf{p}_n defined by (1.8) satisfies

$$\begin{cases} \mathbf{p}_{2k} = (R_2 R_1)^k \mathbf{c}_2 = -(R_1 R_2)^k \mathbf{c}_1, \\ \mathbf{p}_{2k+1} = -R_1 (R_2 R_1)^k \mathbf{c}_2 = R_2 (R_1 R_2)^k \mathbf{c}_1. \end{cases} \quad (2.2)$$

Define the function h_1 by

$$h_1(\mathbf{x}) := \sum_{k=0}^{\infty} \left(\frac{q_{2k}}{|\mathbf{x} + \mathbf{p}_{2k}|} - \frac{q_{2k+1}}{|\mathbf{x} - \mathbf{p}_{2k+1}|} \right). \quad (2.3)$$

Then h_1 is harmonic in $\mathbb{R}^3 \setminus \overline{D_1 \cup D_2}$. Since the circle of Apollonius implies

$$|\mathbf{y} - \mathbf{c}_j| |\mathbf{x} - R_j(\mathbf{y})| = |\mathbf{x} - \mathbf{y}| \quad \text{for } |\mathbf{y} - \mathbf{c}_j| > 1, \mathbf{x} \in \partial D_j, j = 1, 2, \quad (2.4)$$

we have

$$\frac{q_{2k+1}}{|\mathbf{x} - \mathbf{p}_{2k+1}|} = \frac{q_{2k+1}}{|\mathbf{c}_1 - \mathbf{p}_{2k+1}|} \frac{1}{|\mathbf{x} - R_1(\mathbf{p}_{2k+1})|} = \frac{q_{2k+2}}{|\mathbf{x} + \mathbf{p}_{2k+2}|}$$

if $\mathbf{x} \in \partial D_1$, and

$$\frac{q_{2k}}{|\mathbf{x} + \mathbf{p}_{2k}|} = \frac{q_{2k}}{|\mathbf{c}_2 + \mathbf{p}_{2k}|} \frac{1}{|\mathbf{x} - R_2(-\mathbf{p}_{2k})|} = \frac{q_{2k+1}}{|\mathbf{x} - \mathbf{p}_{2k+1}|}$$

if $\mathbf{x} \in \partial D_2$. So we have

$$h_1|_{\partial D_1} = 1, \quad h_1|_{\partial D_2} = 0. \quad (2.5)$$

Moreover, since

$$\frac{1}{4\pi} \int_{\partial D_j} \frac{\partial}{\partial \nu_{\mathbf{x}}} \frac{1}{|\mathbf{x} - \mathbf{y}|} d\sigma(\mathbf{x}) = \begin{cases} -1 & \text{if } \mathbf{y} \in D_j, \\ 0 & \text{if } \mathbf{y} \notin D_j, \end{cases}$$

we have

$$\frac{1}{4\pi} \int_{\partial D_1} \frac{\partial h_1}{\partial \nu} d\sigma = - \sum_{k=0}^{\infty} q_{2k}, \quad \frac{1}{4\pi} \int_{\partial D_2} \frac{\partial h_1}{\partial \nu} d\sigma = \sum_{k=0}^{\infty} q_{2k+1}. \quad (2.6)$$

Define h_2 by

$$h_2(\mathbf{x}) := \sum_{k=0}^{\infty} \left(\frac{q_{2k}}{|\mathbf{x} - \mathbf{p}_{2k}|} - \frac{q_{2k+1}}{|\mathbf{x} + \mathbf{p}_{2k+1}|} \right). \quad (2.7)$$

Then h_2 is harmonic in $\mathbb{R}^3 \setminus \overline{D_1 \cup D_2}$, and one can show similarly that

$$h_2|_{\partial D_1} = 0, \quad h_2|_{\partial D_2} = 1, \quad (2.8)$$

and

$$\frac{1}{4\pi} \int_{\partial D_1} \frac{\partial h_2}{\partial \nu} d\sigma = \sum_{k=0}^{\infty} q_{2k+1}, \quad \frac{1}{4\pi} \int_{\partial D_2} \frac{\partial h_2}{\partial \nu} d\sigma = - \sum_{k=0}^{\infty} q_{2k}. \quad (2.9)$$

It then follows from (2.5), (2.6), (2.8), and (2.9) that the solution to (1.19) is given by

$$h(\mathbf{x}) := - \frac{1}{4\pi \sum_{n=0}^{\infty} q_n} \left(h_1(\mathbf{x}) - h_2(\mathbf{x}) \right) = \frac{1}{4\pi \sum_{n=0}^{\infty} q_n} \sum_{n=0}^{\infty} q_n \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right). \quad (2.10)$$

Thus we have (1.22). We also have

$$h|_{\partial D_2} - h|_{\partial D_1} = \frac{2}{4\pi \sum_{n=0}^{\infty} q_n}. \quad (2.11)$$

In the next section we derive fine properties of the sequences \mathbf{p}_n and q_n , which are used in deriving an asymptotic formula for h .

3 Properties of the sequences \mathbf{p}_n and q_n

Let $\mathbf{p} = (p, 0, 0)$ be the fixed point of the combined reflection $R_2 R_1$ as before. Then one can easily see that p satisfies

$$p = -\frac{1}{1 + \delta + p} + 1 + \delta,$$

so that

$$p = \sqrt{2\delta} + O(\delta) \quad \text{as } \delta \rightarrow 0. \quad (3.1)$$

Let $\mathbf{p}_n = (p_n, 0, 0)$. Then, $p_0 = 1 + \delta$ and p_n satisfies the recursive relations

$$p_{n+1} = -\frac{1}{1 + \delta + p_n} + 1 + \delta, \quad n = 0, 1, \dots \quad (3.2)$$

One can further see that

$$p_n = p \left(\frac{2}{A^{n+1} - 1} + 1 \right) = p \left(\frac{A^{n+1} + 1}{A^{n+1} - 1} \right), \quad (3.3)$$

where

$$A := \frac{1 + \delta + p}{1 + \delta - p}. \quad (3.4)$$

Note that

$$A = 1 + 2p + O(\delta) = 1 + 2\sqrt{2\delta} + O(\delta). \quad (3.5)$$

In particular, the sequence p_n is decreasing and converges to p as $n \rightarrow \infty$.

Since

$$\mu_n = \frac{1}{|\mathbf{c}_1 - \mathbf{p}_n|} = \frac{1}{1 + \delta + p_n} = (1 + \delta - p_{n+1}), \quad (3.6)$$

we have

$$q_{n+1} = \mu_n q_n = \frac{1}{1 + \delta + p_n} q_n = (1 + \delta - p_{n+1}) q_n. \quad (3.7)$$

For a given $\delta > 0$, let $N_0 = N_0(\delta)$, $N = N(\delta)$ and $N_1 = N_1(\delta)$ be as follows:

$$N_0(\delta) = \lfloor \log \delta \rfloor, \quad N(\delta) = \left\lceil \frac{1}{\sqrt{\delta}} \right\rceil, \quad N_1(\delta) = \left\lceil \frac{1}{\delta |\log \delta|} \right\rceil. \quad (3.8)$$

Here $\lfloor \cdot \rfloor$ is the Gaussian bracket. We use this notation for the rest of this paper. Since δ is sufficiently small, we have

$$N_0(\delta) \ll N(\delta) \ll N_1(\delta).$$

The following lemma was obtained in [13].

Lemma 3.1 *There is a constant C independent of δ such that*

$$\left| p_n - \frac{1}{n+1} \right| + \left| q_n - \frac{1}{n+1} \right| \leq C\sqrt{\delta} \quad (3.9)$$

and

$$|p_n - p_{n+1}| < \frac{C}{n^2} \quad (3.10)$$

for $n \leq N(\delta)$.

We prove the following lemma.

Lemma 3.2 *Let $N = N(\delta)$ and $N_1 = N_1(\delta)$ as before.*

(i) *There is a positive C independent of δ such that*

$$\left| \sum_{n=0}^{\infty} q_n - \sum_{n=1}^N \frac{1}{n} \right| \leq C \quad \text{and} \quad \sum_{n=N}^{\infty} q_n \leq C. \quad (3.11)$$

(ii) *$p_n - p \geq 2\sqrt{\delta}A^{-n}$ for all n .*

(iii) *There is a constant C such that*

$$p_n - p \geq \frac{C}{n}$$

for all $n \leq N$.

(iv) *$0 < p_{N_1} - p \leq e^{-1/(\sqrt{\delta}|\log \delta|)}$.*

Proof. Since p_n decays to p , we have from (3.7)

$$q_n \leq (1 + \delta - p)^{n-m} q_m \quad \text{for all } n \geq m \geq 1. \quad (3.12)$$

So, it follows from (3.9) that

$$\sum_{n=N}^{\infty} q_n \leq \sum_{n=N}^{\infty} q_N (1 + \delta - p)^{n-N} \leq \left(C\sqrt{\delta} + \frac{1}{N+1} \right) \sum_{n=N}^{\infty} (1 + \delta - p)^{n-N} \leq C,$$

and

$$\left| \sum_{n=0}^{\infty} q_n - \sum_{n=1}^N \frac{1}{n} \right| \leq C N\sqrt{\delta} + \sum_{n=N}^{\infty} q_n \leq C.$$

This proves (i).

We have from (3.3) that for each $n \in \mathbb{N}$,

$$p_n - p = \frac{2p}{A^{n+1} - 1}. \quad (3.13)$$

So, (ii) follows from (3.1).

Now, suppose that $n \leq N$. Since $A \leq 1 + 3p$, using the inequality

$$(1 + s)^n \leq 1 + ns + \frac{1}{2}n^2s^2(1 + s)^n$$

which holds for all $s > 0$, we obtain

$$A^n \leq (1 + 3p)^n \leq 1 + 3np + \frac{9}{2}n^2p^2(1 + 3p)^n. \quad (3.14)$$

Since $np \leq Np \leq 2$ and $(1 + t)^{1/t}$ increases to e as $t \rightarrow 0+$, we have

$$(1 + 3p)^n \leq \left((1 + 3p)^{\frac{1}{3p}} \right)^{3np} \leq e^6,$$

and hence, from the second inequality in (3.14)

$$A^n \leq 1 + Cnp$$

for some constant C independent of $n \leq N$ and δ . We then infer from (3.3) that

$$p_n - p = \frac{2p}{A^{n+1} - 1} \geq \frac{1}{Cn}, \quad n \leq N(\delta).$$

Now, if $n = N_1$, then we have

$$\log(A^n) = n \log A \geq \frac{n(A-1)}{2} \geq \frac{1}{\sqrt{\delta} |\log \delta|},$$

and hence

$$A^n \geq e^{\frac{1}{\sqrt{\delta} |\log \delta|}}.$$

Now (iv) follows from (3.13). This completes the proof. \square

Lemma 3.2 (i) yields

$$\sum_{n=0}^{\infty} q_n = \frac{1}{2} |\log \delta| + O(1). \quad (3.15)$$

The following lemma provides the finer properties of p_n and q_n that are crucial in proving the main result of this paper.

Lemma 3.3 (i) *If $N_0(\delta) \leq n \leq N_1(\delta)$, then*

$$\frac{q_n}{p_n - p_{n+1}} = \frac{1 + O(|\log \delta|^{-1})}{\sqrt{p_n^2 - p^2}} \quad \text{as } \delta \rightarrow 0, \quad (3.16)$$

where $O(|\log \delta|^{-1})$ is independent of n .

(ii) *There are constants C_1 and C_2 such that*

$$q_n \leq C_1 (1 - p + \delta)^{n - N_1} e^{-\frac{C_2}{\sqrt{\delta} |\log \delta|}} \quad (3.17)$$

for all $n \geq N_1 = N_1(\delta)$.

Proof. If $n > m$, then we have from (3.7)

$$\log q_n = - \sum_{j=m}^{n-1} \log(1 + \delta + p_j) + \log q_m.$$

Using the inequality $|\log(1 + t) - t| \leq Ct^2$, we obtain

$$\log q_n = - \sum_{j=m}^{n-1} p_j - \delta(n - m) + \log q_m + E_1,$$

where the error term E_1 satisfies

$$|E_1| \leq C_1 \sum_{j=m}^{n-1} (\delta + p_j)^2 \leq C_2 \sum_{j=m}^{n-1} p_j^2. \quad (3.18)$$

The last inequality above holds since $\delta \ll p < p_j$. Here and in the rest of this proof, E_j 's denote errors to be estimated. We then have from (3.3) that

$$\log \frac{q_n}{q_m} = -(\delta + p)(n - m) - 2p \sum_{j=m+1}^n \frac{1}{A^j - 1} + E_1.$$

Since $\frac{1}{A^j-1} = \frac{A^{-j}}{1-A^{-j}}$ is decreasing in j , we have

$$\begin{aligned} & \left| \sum_{j=m+1}^n \frac{1}{A^j-1} + \frac{1}{\log A} \log \left(\frac{1-A^{-m-1}}{1-A^{-n-1}} \right) \right| \\ &= \left| \sum_{j=m+1}^n \frac{A^{-j}}{1-A^{-j}} - \int_{m+1}^{n+1} \frac{A^{-x}}{1-A^{-x}} dx \right| \leq \frac{A^{-m-1}}{1-A^{-m-1}}. \end{aligned}$$

So we have

$$\log \frac{q_n}{q_m} = -(\delta + p)(n - m) + \frac{2p}{\log A} \log \left(\frac{1-A^{-m-1}}{1-A^{-n-1}} \right) + E_2,$$

where the new error term E_2 satisfies

$$|E_2| \leq C \left(\sum_{j=m}^{n-1} p_j^2 + \frac{pA^{-m-1}}{1-A^{-m-1}} \right). \quad (3.19)$$

One can see from (3.5) that

$$\frac{2p}{\log A} = 1 + E_3, \quad (3.20)$$

where

$$|E_3| \leq C\sqrt{\delta}. \quad (3.21)$$

So, we have

$$\log \frac{q_n}{q_m} = -(\delta + p)(n - m) + (1 + E_3) \log \left(\frac{1-A^{-m-1}}{1-A^{-n-1}} \right) + E_2,$$

which in turn implies

$$q_n = q_m e^{-pn} \left(\frac{1-A^{-m-1}}{1-A^{-n-1}} \right) e^{E_4}, \quad (3.22)$$

where

$$E_4 := pm - \delta(n - m) + E_2 + E_3 \log \left(\frac{1-A^{-m-1}}{1-A^{-n-1}} \right). \quad (3.23)$$

Note that

$$p_n - p_{n+1} = (p_{n+1} - p) \frac{A-1}{1-A^{-n-1}},$$

so that

$$\frac{q_n}{p_n - p_{n+1}} = \frac{q_m e^{-pn}}{p_{n+1} - p} \frac{1-A^{-m-1}}{A-1} e^{E_4}. \quad (3.24)$$

Since $p_n/p = (A^{n+1} + 1)/(A^{n+1} - 1)$, we have

$$(n+1) \log A = \log \left(\frac{p_n + p}{p_n - p} \right),$$

and, since $\log A = 2p + O(\delta)$, it follows that

$$pn = \frac{1}{2} \log \left(\frac{p_n + p}{p_n - p} \right) + E_5,$$

where

$$|E_5| \leq C(n\delta + \sqrt{\delta}). \quad (3.25)$$

We then obtain from (3.24)

$$\begin{aligned}\frac{q_n}{p_n - p_{n+1}} &= \sqrt{\frac{p_n - p}{p_n + p}} \frac{1}{p_{n+1} - p} q_m \frac{1 - A^{-m-1}}{A - 1} e^{E_4 - E_5} \\ &= \frac{1}{\sqrt{p_n^2 - p^2}} \frac{p_n - p}{p_{n+1} - p} q_m \frac{1 - A^{-m-1}}{A - 1} e^{E_4 - E_5}.\end{aligned}\quad (3.26)$$

Suppose now that $m = N_0 - 1$ and $m < n \leq N_1$. Then we have $E_5 = O(|\log \delta|^{-1})$. We will show that

$$\frac{p_n - p}{p_{n+1} - p} = 1 + O(|\log \delta|^{-1}), \quad (3.27)$$

$$q_m \frac{1 - A^{-m-1}}{A - 1} = 1 + O(|\log \delta|^{-1}), \quad (3.28)$$

$$E_4 = O(|\log \delta|^{-1}). \quad (3.29)$$

Once we have these estimates, then (i) follows from (3.26).

To prove (3.27), we first observe that

$$\frac{p_n - p}{p_{n+1} - p} = \frac{A^{n+2} - 1}{A^{n+1} - 1} = A \left(1 + \frac{1}{A + A^2 + \dots + A^{n+1}} \right).$$

Since $A > 1$, $n \geq |\log \delta|$ and $A = 1 + O(\sqrt{\delta})$, we have

$$\frac{p_n - p}{p_{n+1} - p} = (1 + O(\sqrt{\delta}))(1 + O(|\log \delta|^{-1})) = 1 + O(|\log \delta|^{-1}).$$

To prove (3.28), we use inequalities

$$(m+1)s - \frac{1}{2}m(m+1)s^2 \leq 1 - (1-s)^{m+1} \leq (m+1)s$$

which hold for all $s \in [0, 1]$. Since $A^{-1} = 1 - 2p + O(\delta)$, we have

$$(m+1)(2p + O(\delta)) - \frac{1}{2}m(m+1)(2p + O(\delta))^2 \leq 1 - A^{-m-1} \leq (m+1)(2p + O(\delta)).$$

Since $m = O(|\log \delta|)$ and $p = O(\sqrt{\delta})$, we have

$$1 - A^{-m-1} = 2(m+1)p + O(\delta |\log \delta|^2). \quad (3.30)$$

Note that

$$\frac{1}{A - 1} = \frac{1}{2p + O(\delta)} = \frac{1}{2p} + O(1).$$

Since $q_m = \frac{1}{m+1} + O(\sqrt{\delta})$ by Lemma 3.1 and $m = N_0 - 1$, we infer that

$$\begin{aligned}q_m \frac{1 - A^{-m-1}}{A - 1} &= \left(\frac{1}{m+1} + O(\sqrt{\delta}) \right) (2(m+1)p + O(\delta |\log \delta|^2)) \left(\frac{1}{2p} + O(1) \right) \\ &= 1 + O(|\log \delta|^{-1}).\end{aligned}$$

So, (3.28) is proved.

To prove (3.29), we first estimate E_2 . We have from (3.9) that

$$\begin{aligned} \sum_{j=m+1}^n p_j^2 &\leq C \sum_{j=N_0}^N p_j^2 + \sum_{j=N+1}^{N_1} p_j^2 \\ &\leq C \sum_{j=N_0}^N \frac{1}{j^2} + \sum_N^{N_1} p_N^2 \\ &\leq C \left(\frac{1}{N_0} + p_N^2 N_1 \right) = O(|\log \delta|^{-1}). \end{aligned}$$

On the other hand, it follows from (3.30) that

$$\frac{pA^{-m-1}}{1-A^{-m-1}} = \frac{p + O(\delta|\log \delta|)}{2(m+1)p + O(\delta|\log \delta|^2)} = \frac{1}{2(m+1)} \left(1 + O(\sqrt{\delta}|\log \delta|) \right) = O(|\log \delta|^{-1}).$$

So we infer from (3.19) that

$$E_2 = O(|\log \delta|^{-1}). \quad (3.31)$$

Since $p = O(\sqrt{\delta})$, we obtain from (3.30) that

$$1 \geq \frac{1 - A^{-m-1}}{1 - A^{-n-1}} \geq 1 - A^{-m-1} \geq C\sqrt{\delta}|\log \delta|.$$

We then infer from (3.21) that

$$\left| E_3 \log \left(\frac{1 - A^{-m-1}}{1 - A^{-n-1}} \right) \right| \leq C\sqrt{\delta}|\log \delta|. \quad (3.32)$$

Thus we have from (3.23), (3.31) and (3.32) that

$$\begin{aligned} |E_4| &\leq |pm| + \delta(n-m) + |E_2| + \left| E_3 \log \left(\frac{1 - A^{-m-1}}{1 - A^{-n-1}} \right) \right| \\ &\leq C \left(\sqrt{\delta}|\log \delta| + |\log \delta|^{-1} \right) \leq C|\log \delta|^{-1}, \end{aligned}$$

so (3.29) is proved.

We have from (3.22) that

$$q_{N_1} = q_m e^{-pN_1} \left(\frac{1 - A^{-m-1}}{1 - A^{-N_1-1}} \right) e^{E_4}.$$

So, it follows from (3.29) that

$$q_{N_1} \leq C_1 e^{-\frac{C_2}{\sqrt{\delta}|\log \delta|}}$$

for some constants C_1 and C_2 . Now, (ii) follows from (3.12). This completes the proof. \square

4 Asymptotic behavior of the singular function

Let

$$R_\delta := \{ \mathbf{x} \in \mathbb{R}^3 \setminus (D_1 \cup D_2) \mid \rho(\mathbf{x}) \leq |\log \delta|^{-2} \}, \quad (4.1)$$

where ρ is given by (2.1). Note that R_δ is a narrow region in between D_1 and D_2 . Let

$$v(\mathbf{x}) := \left(4\pi \sum_{n=0}^{\infty} q_n \right) h(\mathbf{x}) = \sum_{n=0}^{\infty} q_n \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right). \quad (4.2)$$

In this section we investigate the asymptotic behavior of $\nabla v(\mathbf{x})$ in the region R_δ . We obtain the following proposition.

Proposition 4.1 For $\mathbf{x} = (x, y, z) \in R_\delta$, we have

$$\nabla v(\mathbf{x}) = \frac{2}{2\delta + \rho(\mathbf{x})^2} ((1, 0, 0) + O(|\log \delta|^{-1})). \quad (4.3)$$

It turns out that $|\partial_y v(\mathbf{x})|$ and $|\partial_z v(\mathbf{x})|$ can be estimated without much difficulty. In fact, we obtain the following lemma whose proof is given in Subsection 4.1.

Lemma 4.2 For $\mathbf{x} = (x, y, z) \in R_\delta$, we have

$$|\partial_y v(\mathbf{x})| + |\partial_z v(\mathbf{x})| \leq \frac{C}{\sqrt{\delta} + \rho(\mathbf{x})} \left(1 + \log \left(1 + \frac{\rho(\mathbf{x})^2}{\delta} \right) \right) \quad (4.4)$$

for some constant C independent of δ .

Estimates of $\partial_x v(\mathbf{x})$, especially those terms for $N_0 \leq n \leq N_1$, are quite involved. Based on Lemma 3.3 (i) we compare v which is given as an infinite series with the integral defined by

$$v_0(\mathbf{x}) := \int_p^1 \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} dt, \quad (4.5)$$

where $(p, 0, 0)$ be the fixed point of the combined reflections $R_2 R_1$. We obtain the following lemmas whose proofs are given in Subsection 4.2 and 4.3, respectively.

Lemma 4.3 For $\mathbf{x} = (x, y, z) \in R_\delta$, we have

$$\partial_x v_0(\mathbf{x}) = \frac{2}{2\delta + \rho(\mathbf{x})^2} (1 + O(|\log \delta|^{-1})). \quad (4.6)$$

Lemma 4.4 For $\mathbf{x} = (x, y, z) \in R_\delta$, we have

$$\partial_x v(\mathbf{x}) = \partial_x v_0(\mathbf{x}) (1 + O(|\log \delta|^{-1})). \quad (4.7)$$

Proposition 4.1 is an immediate consequence of above lemmas.

4.1 Proof of Lemma 4.2

We first observe that if $\mathbf{x} = (x, y, z) \in R_\delta$, then $|x| \leq 1 + \delta - \sqrt{1 - y^2 - z^2}$ and $\rho \leq |\log \delta|^{-2}$, and hence

$$|x| \leq \delta + \rho(\mathbf{x})^2. \quad (4.8)$$

Using notation $\rho = \rho(\mathbf{x})$, v can be expressed as

$$v(\mathbf{x}) = \sum_{n=0}^{\infty} q_n \left(\frac{1}{\sqrt{(x - p_n)^2 + \rho^2}} - \frac{1}{\sqrt{(x + p_n)^2 + \rho^2}} \right). \quad (4.9)$$

So, it suffices to estimate $|\partial_\rho v|$. Note that

$$\begin{aligned} \partial_\rho \left(\frac{1}{\sqrt{(x - p_n)^2 + \rho^2}} - \frac{1}{\sqrt{(x + p_n)^2 + \rho^2}} \right) &= -\frac{\rho}{[(x - p_n)^2 + \rho^2]^{3/2}} + \frac{\rho}{[(x + p_n)^2 + \rho^2]^{3/2}} \\ &= 3\rho \int_{-x}^x \frac{t - p_n}{[(t - p_n)^2 + \rho^2]^{5/2}} dt. \end{aligned}$$

Therefore, we have

$$\left| \partial_\rho \left(\frac{1}{\sqrt{(x - p_n)^2 + \rho^2}} - \frac{1}{\sqrt{(x + p_n)^2 + \rho^2}} \right) \right| \leq 3\rho \int_{-x}^x \frac{1}{[(t - p_n)^2 + \rho^2]^2} dt.$$

By (4.8) we have

$$(t - p_n)^2 + \rho^2 \geq C(\rho^2 + p_n^2) \quad (4.10)$$

for some constant C . It then follows that

$$\left| \partial_\rho \left(\frac{1}{\sqrt{(x - p_n)^2 + \rho^2}} - \frac{1}{\sqrt{(x + p_n)^2 + \rho^2}} \right) \right| \leq C \frac{\rho(\rho^2 + \delta)}{\rho^4 + p_n^4} \leq C \frac{\rho}{\rho^2 + p_n^2}.$$

So we have

$$|\partial_\rho v(\mathbf{x})| \leq C \sum_{n=0}^{\infty} \frac{\rho q_n}{\rho^2 + p_n^2}.$$

Let $N = N(\delta)$. Using Lemma 3.1, we have

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{\rho q_n}{\rho^2 + p_n^2} &\leq C \sum_{n=1}^N \frac{\rho}{(1/n^2 + \rho^2)} \frac{1}{n} \\ &\leq C \sum_{n=1}^N \frac{\rho n}{1 + \rho^2 n^2} \leq C \left(1 + \int_1^{1/\sqrt{\delta}} \frac{\rho s}{1 + \rho^2 s^2} ds \right) \\ &\leq C \frac{1}{\rho + \sqrt{\delta}} \left(1 + \log \left(1 + \frac{\rho^2}{\delta} \right) \right). \end{aligned}$$

If $n \geq N$, then $q_n = O(\sqrt{\delta})$, and thus we have from (3.12)

$$\sum_{n=N}^{\infty} \frac{\rho q_n}{\rho^2 + p_n^2} \leq C \sum_{n=N}^{\infty} \frac{\rho}{\rho^2 + \delta} \sqrt{\delta} (1 + \delta - p)^{n-N} \leq \frac{C}{\rho + \sqrt{\delta}}.$$

This completes the proof. \square

4.2 Proof of Lemma 4.3

Let

$$\begin{aligned} \partial_x v_0(\mathbf{x}) &= \int_p^{|\log \delta|^{-1}} + \int_{|\log \delta|^{-1}}^1 \partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} dt \\ &:= I + II. \end{aligned}$$

If $|\log \delta|^{-1} \leq t \leq 1$, then $|\mathbf{x} \pm (t, 0, 0)| \geq Ct$ for some constant C and for all $\mathbf{x} \in \mathbb{R}^3 \setminus (D_1 \cup D_2)$. Since $p = O(\sqrt{\delta})$, we also have $\sqrt{t^2 - p^2} \geq Ct$. Thus we have

$$|II| \leq C \int_{|\log \delta|^{-1}}^1 \frac{1}{t^3} dt \leq C |\log \delta|^2. \quad (4.11)$$

Suppose now that $p \leq t \leq |\log \delta|^{-1}$. Using (4.8) and the fact that $p = O(\sqrt{\delta})$ again, we have for all $\mathbf{x} \in R_\delta$

$$|tx| \leq t(\rho^2 + \delta) \leq \frac{C}{|\log \delta|} (t^2 + \rho^2), \quad (4.12)$$

$$|x|^2 \leq (\rho^2 + \delta)^2 \leq \frac{C}{|\log \delta|} (t^2 + \rho^2) \quad (4.13)$$

for some constant C independent of δ . Thus, we have

$$\begin{aligned} \frac{1}{|\mathbf{x} \pm (t, 0, 0)|^3} &= \frac{1}{((x \pm t)^2 + \rho^2)^{3/2}} = \frac{1}{(t^2 + \rho^2 \pm 2xt + x^2)^{3/2}} \\ &= \frac{1}{(t^2 + \rho^2)^{3/2}} (1 + O(|\log \delta|^{-1})). \end{aligned} \quad (4.14)$$

From the mean value property, we have

$$\begin{aligned} &\left| \frac{-1}{|\mathbf{x} - (t, 0, 0)|^3} + \frac{1}{|\mathbf{x} + (t, 0, 0)|^3} \right| \\ &= \left| \frac{1}{((t^2 + x^2 + \rho^2) - 2xt)^{3/2}} - \frac{1}{((t^2 + x^2 + \rho^2) + 2xt)^{3/2}} \right| \\ &\leq \frac{6|xt|}{|(t^2 + x^2 + \rho^2) - |2xt||^{5/2}}. \end{aligned} \quad (4.15)$$

It then follows from (4.12) and (4.13) that

$$\left| x \left(\frac{-1}{|\mathbf{x} - (t, 0, 0)|^3} + \frac{1}{|\mathbf{x} + (t, 0, 0)|^3} \right) \right| \leq C |\log \delta|^{-1} \frac{t}{(t^2 + \rho(\mathbf{x})^2)^{3/2}} \quad (4.16)$$

for some constant C independent of δ .

Since

$$\begin{aligned} &\partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \\ &= -\frac{x-t}{|\mathbf{x} - (t, 0, 0)|^3} + \frac{x+t}{|\mathbf{x} + (t, 0, 0)|^3} \\ &= t \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|^3} + \frac{1}{|\mathbf{x} + (t, 0, 0)|^3} \right) + x \left(\frac{-1}{|\mathbf{x} - (t, 0, 0)|^3} + \frac{1}{|\mathbf{x} + (t, 0, 0)|^3} \right), \end{aligned} \quad (4.17)$$

we obtain from (4.14) and (4.16)

$$\partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) = \frac{t}{(t^2 + \rho^2)^{3/2}} (2 + O(|\log \delta|^{-1})).$$

It then follows that

$$I = (2 + O(|\log \delta|^{-1})) \int_p^{|\log \delta|^{-1}} \frac{t}{(t^2 + \rho^2)^{3/2}} \frac{1}{\sqrt{t^2 - p^2}} dt.$$

Using the substitution $t = \sqrt{t^2 - p^2}$, one can see that

$$\int_p^{|\log \delta|^{-1}} \frac{t}{(t^2 + \rho^2)^{3/2}} \frac{1}{\sqrt{t^2 - p^2}} dt = \frac{1}{p^2 + \rho^2} \left(\frac{|\log \delta|^{-2} - p^2}{|\log \delta|^{-2} + \rho^2} \right)^{1/2}.$$

Since $p = \sqrt{2\delta} + O(\delta)$ and $\rho \leq |\log \delta|^{-2}$, we have

$$\frac{1}{p^2 + \rho^2} \left(\frac{|\log \delta|^{-2} - p^2}{|\log \delta|^{-2} + \rho^2} \right)^{1/2} = \frac{1}{2\delta + \rho^2} (1 + O(|\log \delta|^{-1})),$$

and hence

$$I = \frac{2}{2\delta + \rho^2} (1 + O(|\log \delta|^{-1})),$$

which together with (4.11) yields

$$\partial_x v_0(\mathbf{x}) = \frac{2}{2\delta + \rho^2} (1 + O(|\log \delta|^{-1})) + O(|\log \delta|^2).$$

Since $\rho \leq |\log \delta|^{-2}$, the above formula can be written as

$$\partial_x v_0(\mathbf{x}) = \frac{2}{2\delta + \rho^2} (1 + O(|\log \delta|^{-1})).$$

This completes the proof. □

4.3 Proof of Lemma 4.4

Let $N_0 = \lfloor |\log \delta| \rfloor$ and $N_1 = \lfloor \frac{1}{\delta |\log \delta|} \rfloor$ as before. Let

$$\begin{aligned} \partial_x v(\mathbf{x}) &= \sum_{n=0}^{N_0-1} + \sum_{n=N_0}^{N_1-1} + \sum_{n=N_1}^{\infty} \partial_x \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right) q_n \\ &:= S_1(\mathbf{x}) + S_2(\mathbf{x}) + S_3(\mathbf{x}), \end{aligned}$$

and

$$\begin{aligned} \partial_x v_0(\mathbf{x}) &= \int_{p_{N_0}}^1 + \int_{p_{N_1}}^{p_{N_0}} + \int_p^{p_{N_1}} \partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} dt \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

We first estimate S_1 , S_3 , I_1 , and I_3 . There is a constant $C > 0$ independent of n such that $|\mathbf{x} \pm \mathbf{p}_n| \geq Cp_n$ for all $\mathbf{x} \in R_\delta$. So we have from (3.9) that

$$\begin{aligned} |S_1(\mathbf{x})| &\leq \sum_{n=0}^{N_0-1} \left| \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right) q_n \right| \\ &\leq C \sum_{n=1}^{N_0-1} n^2 \frac{1}{n} \leq C |\log \delta|^2. \end{aligned}$$

We also have from Lemma 3.3 (ii) that

$$\begin{aligned} |S_3(\mathbf{x})| &\leq \sum_{n=N_1}^{\infty} \left| \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right) q_n \right| \leq \sum_{n=N_1}^{\infty} \frac{1}{p^2} q_n \\ &\leq C \sum_{n=N_1}^{\infty} \frac{1}{\delta} (1 - p + \delta)^{n-N_1} e^{-C_2 \frac{1}{\sqrt{\delta} |\log \delta|}} \leq C. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |I_1(\mathbf{x})| &\leq \int_{p_{N_0}}^1 \left| \nabla \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} \right| dt \\ &\leq C \int_{|\log \delta|^{-1}}^1 \frac{1}{t^3} dt \leq C |\log \delta|^2, \end{aligned}$$

and by Lemma 3.2 (iv)

$$\begin{aligned} |I_3(\mathbf{x})| &\leq \left| \int_p^{p_{N_1}} \nabla \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} dt \right| \\ &\leq C \int_p^{p_{N_1}} \frac{1}{\delta^{5/4}} \frac{1}{\sqrt{t - p}} dt \leq C \frac{1}{\delta^{5/4} e^{1/(2\sqrt{\delta} |\log \delta|)}} \leq C. \end{aligned}$$

So far, we showed that

$$|S_1| + |S_3| + |I_1| + |I_3| \leq C |\log \delta|^2. \quad (4.18)$$

We set

$$\tilde{S}_2(\mathbf{x}) = \sum_{n=N_0}^{N_1-1} \partial_x \left(\frac{1}{|\mathbf{x} - \mathbf{p}_n|} - \frac{1}{|\mathbf{x} + \mathbf{p}_n|} \right) \frac{p_n - p_{n+1}}{\sqrt{p_n^2 - p^2}},$$

and shall prove

$$|\tilde{S}_2(\mathbf{x}) - I_2(\mathbf{x})| \leq C \frac{1}{\sqrt{\delta} + \rho}. \quad (4.19)$$

Let us first show that Lemma 4.4 follows from (4.18) and (4.19). We observe from Lemma 3.3 (i) that

$$S_2(\mathbf{x}) = \tilde{S}_2(\mathbf{x}) (1 + O(|\log \delta|^{-1})).$$

So we have

$$\begin{aligned} \partial_x v &= S_1 + S_2 + S_3 \\ &= \tilde{S}_2 (1 + O(|\log \delta|^{-1})) + S_1 + S_3 \\ &= I_2 (1 + O(|\log \delta|^{-1})) + (\tilde{S}_2 - I_2) (1 + O(|\log \delta|^{-1})) + S_1 + S_3 \\ &= \partial_x v_0 (1 + O(|\log \delta|^{-1})) + R, \end{aligned}$$

where

$$R = -(I_1 + I_3) (1 + O(|\log \delta|^{-1})) + (\tilde{S}_2 - I_2) (1 + O(|\log \delta|^{-1})) + S_1 + S_3.$$

Since $\rho \leq |\log \delta|^{-2}$, one can see from (4.18) and (4.19) that

$$|R| \leq C \left(|\log \delta|^2 + \frac{1}{\sqrt{\delta} + \rho} \right).$$

Thanks to (4.6), we now have

$$\partial_x v = \partial_x v_0 (1 + O(|\log \delta|^{-1})) + R = \partial_x v_0 (1 + O(|\log \delta|^{-1})),$$

which we aim to prove.

The rest of this subsection is devoted to the proof of (4.19). For $N_0 \leq n \leq N_1$, let

$$\begin{aligned} \gamma_n(\mathbf{x}) &:= \left| \partial_x \left(\frac{1}{|\mathbf{x} - (p_n, 0, 0)|} - \frac{1}{|\mathbf{x} + (p_n, 0, 0)|} \right) \frac{p_n - p_{n+1}}{\sqrt{p_n^2 - p^2}} \right. \\ &\quad \left. - \int_{p_{n+1}}^{p_n} \partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}} dt \right|. \end{aligned}$$

Let

$$f(t) := \partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \frac{1}{\sqrt{t^2 - p^2}}.$$

By the mean-value property there is $t_n \in [p_{n+1}, p_n]$ such that

$$f(p_n)(p_n - p_{n+1}) - \int_{p_{n+1}}^{p_n} f(t)dt = \frac{f'(t_n)}{2}(p_n - p_{n+1})^2.$$

So we have

$$\begin{aligned} \gamma_n(\mathbf{x}) &\leq \frac{1}{2} \left| \partial_t \partial_x \left(\frac{1}{|\mathbf{x} - (t, 0, 0)|} - \frac{1}{|\mathbf{x} + (t, 0, 0)|} \right) \right|_{t=t_n} \left| \frac{1}{\sqrt{t_n^2 - p^2}} (p_n - p_{n+1})^2 \right. \\ &\quad \left. + \frac{1}{2} \left| \partial_x \left(\frac{1}{|\mathbf{x} - (t_n, 0, 0)|} - \frac{1}{|\mathbf{x} + (t_n, 0, 0)|} \right) \right| \frac{t_n}{(t_n^2 - p^2)^{3/2}} (p_n - p_{n+1})^2 \right. \\ &\quad \left. := \frac{1}{2} (\gamma_{n1}(\mathbf{x}) + \gamma_{n2}(\mathbf{x})). \right. \end{aligned}$$

Using (4.8), one can show that

$$|\mathbf{x} \pm (t_n, 0, 0)|^2 \geq C(\rho^2 + t_n^2), \quad \mathbf{x} \in R_\delta \quad (4.20)$$

for some constant independent of n . So we have

$$\gamma_{n1} \leq \frac{C}{\rho^3 + t_n^3} \frac{1}{\sqrt{t_n^2 - p^2}} (p_n - p_{n+1})^2 \quad (4.21)$$

and

$$\gamma_{n2} \leq \frac{C}{\rho^2 + t_n^2} \frac{t_n}{(t_n^2 - p^2)^{3/2}} (p_n - p_{n+1})^2. \quad (4.22)$$

If $n \leq N = \lceil \frac{1}{\sqrt{\delta}} \rceil$, then we have $t_n \approx 1/n$ and $|p_n - p_{n+1}| < C/n^2$ by Lemma 3.1, and $p_n - p \geq C/n$ by Lemma 3.2 (iii). So, we have

$$\sum_{n=N_0}^N \gamma_{n1} \leq C \sum_{n=1}^N \frac{1}{(1/n^3 + \rho^3)\sqrt{1/n^2}} \frac{1}{n^4} \leq C \sum_{n=1}^N \frac{1}{1 + \rho^3 n^3}.$$

Note that if $\rho \leq \sqrt{\delta}$, then

$$\sum_{n=1}^N \frac{1}{1 + \rho^3 n^3} \leq N \leq \frac{1}{\sqrt{\delta}},$$

while if $\rho > \sqrt{\delta}$, then

$$\sum_{n=1}^N \frac{1}{1 + \rho^3 n^3} \leq \frac{1}{\rho^3} \sum_{n=1}^N \frac{1}{n^3} \leq \frac{C}{\rho}.$$

So, we have

$$\sum_{n=N_0}^N \gamma_{n1} \leq \frac{C}{\sqrt{\delta} + \rho}.$$

If $N \leq n \leq N_1$, we have from (3.3) that

$$0 \leq p_n - p_{n+1} = 2p \frac{A^{-n-1}(1 - A^{-1})}{(1 - A^{-n-1})(1 - A^{-n-2})} \leq C_2 \delta A^{-n},$$

and by Lemma 3.2 (ii), $p_n - p \geq C\sqrt{\delta}A^{-n}$. Since $p_n > p$ for all n , we have

$$\begin{aligned}
\sum_{n=N+1}^{N_1-1} \gamma_{n1} &\leq C \sum_{n=N+1}^{N_1-1} \frac{1}{(p^3 + \rho^3)\sqrt{p_{n+1}^2 - p^2}} (p_n - p_{n+1})^2 \\
&\leq C \sum_{n=N+1}^{\infty} \frac{1}{(\delta^{3/2} + \rho^3)\delta^{1/2}A^{-n/2}} \delta^2 A^{-2n} \\
&\leq C \sum_{n=N+1}^{\infty} \frac{1}{\delta^{1/2}(\delta^{3/2} + \rho^3)} \delta^2 A^{-(3/2)n} \\
&\leq C \frac{\delta^{3/2}}{\delta^{3/2} + \rho^3} \frac{1}{\sqrt{\delta}} \leq \frac{C}{\sqrt{\delta} + \rho}.
\end{aligned}$$

Thus, we have

$$\sum_{n=N_0}^{N_1} \gamma_{n1} \leq \frac{C}{\rho + \sqrt{\delta}}.$$

Similarly one can show that

$$\sum_{n=N_0}^{N_1} \gamma_{n2} \leq \frac{C}{\rho + \sqrt{\delta}}.$$

This completes the proof of (4.19). □

5 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. It is helpful to recall that $\epsilon = 2\delta$. We have from (1.20) and (2.11) that

$$\begin{aligned}
u &= \frac{u|_{\partial D_1} - u|_{\partial D_2}}{h|_{\partial D_1} - h|_{\partial D_2}} h + g \\
&= \frac{1}{2}(u|_{\partial D_2} - u|_{\partial D_1}) \left(4\pi \sum_{n=0}^{\infty} q_n\right) h + g \\
&= \frac{1}{2}(u|_{\partial D_2} - u|_{\partial D_1}) v + g.
\end{aligned} \tag{5.1}$$

We emphasize that $|\nabla g|$ is bounded on any bounded subset of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$ regardless of δ as explained in Introduction. Since h is constant on ∂D_1 and ∂D_2 , one can see from (1.21) and (1.22) that

$$\begin{aligned}
u|_{\partial D_2} - u|_{\partial D_1} &= - \int_{\partial(D_1 \cup D_2)} H \frac{\partial h}{\partial \nu} d\sigma \\
&= - \int_{\partial(D_1 \cup D_2)} H \frac{\partial h}{\partial \nu} d\sigma + \int_{\partial(D_1 \cup D_2)} \frac{\partial H}{\partial \nu} h d\sigma \\
&= - \frac{1}{\sum_{n=0}^{\infty} q_n} \sum_{n=0}^{\infty} q_n \int_{\partial(D_1 \cup D_2)} \left[H(\mathbf{x}) \frac{\partial}{\partial \nu} (\Gamma(\mathbf{x} - \mathbf{p}_n) - \Gamma(\mathbf{x} + \mathbf{p}_n)) \right. \\
&\quad \left. - \frac{\partial H}{\partial \nu} (\Gamma(\mathbf{x} - \mathbf{p}_n) - \Gamma(\mathbf{x} + \mathbf{p}_n)) \right] d\sigma \\
&= \frac{C_H^\epsilon}{2 \sum_{n=0}^{\infty} q_n}.
\end{aligned}$$

It then follows from (3.15) that

$$u|_{\partial D_2} - u|_{\partial D_1} = \frac{C_H^\epsilon}{|\log \delta|} (1 + O(|\log \delta|^{-1})), \quad (5.2)$$

where $O(|\log \delta|^{-1})$ is independent of H . So we obtain from (5.1)

$$\nabla u = \frac{C_H^\epsilon}{2|\log \delta|} (1 + O(|\log \delta|^{-1})) \nabla v + \nabla g,$$

and from (4.3)

$$\nabla u(\mathbf{x}) = \frac{C_H^\epsilon}{|\log \delta|(2\delta + \rho(\mathbf{x})^2)} ((1, 0, 0) + O(|\log \delta|^{-1})) + \nabla g(\mathbf{x}), \quad (5.3)$$

and hence (1.12) is proved. This completes the proof. \square

Proof of Theorem 1.2. By (3.9), we have for $n \leq N = N(\delta)$

$$\begin{aligned} & \left| q_n(H(\mathbf{p}_n) - H(-\mathbf{p}_n)) - \frac{1}{n+1} \left(H\left(\frac{1}{n+1}, 0, 0\right) - H\left(-\frac{1}{n+1}, 0, 0\right) \right) \right| \\ & \leq \left| q_n - \frac{1}{n+1} \right| |H(\mathbf{p}_n) - H(-\mathbf{p}_n)| \\ & \quad + \frac{1}{n+1} \left(\left| H(\mathbf{p}_n) - H\left(\frac{1}{n+1}, 0, 0\right) \right| + \left| H(-\mathbf{p}_n) - H\left(-\frac{1}{n+1}, 0, 0\right) \right| \right) \\ & \leq C\sqrt{\delta} \left(\sqrt{\delta} + \frac{1}{n+1} \right) + C\sqrt{\delta} \frac{1}{n+1}. \end{aligned}$$

So we have

$$\left| \sum_{n=0}^{N-1} q_n(H(\mathbf{p}_n) - H(-\mathbf{p}_n)) - \sum_{n=1}^N \frac{1}{n} \left(H\left(\frac{1}{n}, 0, 0\right) - H\left(-\frac{1}{n}, 0, 0\right) \right) \right| \leq C\sqrt{\delta} |\log \delta|.$$

On the other hand, since p_n is decreasing, it follows from (3.15) that

$$\left| \sum_{n=N}^{\infty} q_n(H(\mathbf{p}_n) - H(-\mathbf{p}_n)) \right| \leq Cp_N \sum_{n=N}^{\infty} q_n \leq C\sqrt{\delta} |\log \delta|.$$

We also have

$$\left| \sum_{n=N+1}^{\infty} \frac{1}{n} \left(H\left(\frac{1}{n}, 0, 0\right) - H\left(-\frac{1}{n}, 0, 0\right) \right) \right| \leq C \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq C\sqrt{\delta}.$$

Combining above estimates, we obtain (1.16).

The formula (1.18) is an immediate consequence of (1.12) and (1.16). This completes the proof. \square

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